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WAVY FLOW OF THIN LIQUID FILMS

V. PENEV, V. S. KRYLOV, CH. BOYADJIEV and V. P. VOROTILIN
Institute of General and Inorganic Chemistry. Bulgarian Academy of Sciences. Sofia. Bulgaria

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Abstract— An integral method is developed for solving the system of the Navier–Stokes equations describing the velocity distribution in a laminar liquid film freely flowing under the action of the gravity force along a vertical solid wall. With the aid of this method the characteristic parameters of the regular film flow (the wavelength, the celerity and the amplitude) are calculated. In the linear approximation with respect to the ratio of the amplitude to the mean film thickness, analytical expressions are obtained determining the dependence of the wavelength and the celerity on the Weber number. The parameters of waves with amplitude comparable with the mean film thickness are calculated using a digital computer. In accordance with the available experimental data, it is found that at given values of the surface tension, the density and the viscosity of liquid, the celerity decreases monotonously with the increasing Weber number, the wavelength as a function of the Weber number passing through a minimum. The mechanism of the effect of waves on mass transfer is discussed.

NOMENCLATURE

A, dimensionless amplitude of waves $= (h_{max} - h_0)\, h_0$;
g, acceleration due to gravity;
E, kinetic energy of liquid in a regular wavy film;
E^*, kinetic energy of liquid in a disturbed state;
E', kinetic energy due to perturbation;
$h(x, t)$, local film thickness;
h_0, mean (over the wavelength) film thickness;
h_{max} film thickness in cross sections corresponding to the crests of waves;
n, dimensionless wave number $= 2\pi h_0/\lambda$;
p, pressure;
t, time;
u, x-component of the liquid velocity;
u_0, mean (over the cross section of the film and the wavelength) film velocity;
i, y-component of the liquid velocity;
$\mathbf{v}$, velocity vector with the components u and v;
x, coordinate in the film flow direction;
y, coordinate in the direction perpendicular to the solid wall (counted from the wall);
α, dimensionless phase velocity of wave reduced to the mean velocity u_0;
λ, wavelength;
ν, kinematic viscosity of liquid;
ρ, density of liquid;
ξ, dimensionless longitudinal coordinate in the coordinate system moving along the x-axis with the velocity αu_0, $\xi = (x - \alpha u_0 t)/h_0$;
σ, surface tension;
Re, Reynolds number $= u_0 h_0/\nu$;
We, Weber number $= \rho h_0 u_0^2/\sigma$;
Λ, dimensionless wavelength $= (\lambda/2\pi)(\rho g/\sigma)^{\frac{1}{2}}$.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

THE WAVY regime of flow of thin laminar liquid films driven by gravity to descend along inclined solid surfaces is a subject of extensive experimental and theoretical investigation because this regime is encountered in many types of chemical engineering equipment (for example, in tubular wetted-wall columns, in columns with flat-parallel packing, etc.). By virtue of inducing additional lateral motion of the liquid in the films and increasing the interphase area the wave formation stimulates augmentation of mass transfer between a film and an adjacent gas phase. According to the available experimental data [1] the increase of the mass transfer coefficient caused by wave formation in the processes of film rectification and film absorption may be as large as 50 per cent and more.

The first attempt to develop a quantitative theory of wavy motion of liquid films was made by Kapitza [2] who based his considerations on the assumption of smallness (compared with unity) of the mean film thickness to wavelength ratio. In accordance with this assumption the system of the boundary layer equations was used for description of the velocity distribution in a film and a non-damped periodic solution was obtained in the form of half-parabolic profile with the mean velocity depending on the coordinate in the direction of flow. With accuracy up to the terms of the second order of smallness with respect to the wave amplitude to mean film thickness ratio an expression for a shape free film surface was obtained and the main parameters of wavy flow (the wave length, the celerity and the amplitude) were calculated using a hypothesis about the minimum of viscous dissipation of the kinetic energy.

The theoretical results obtained by Kapitza appeared to be in quite satisfactory agreement with experimental data corresponding to small liquid flow rates [3]. With the increasing flow rate appreciable deviations from the theory began to be observed. Namely, the wavelength decrease monotonously with the increasing flow rate, passes through a minimum and begins to increase [4]. Besides, the observed celerity generally decreases with the increasing flow rate, in contradiction to the Kapitza's theory which predicts constant celerity.

Among the papers on a theory of wavy film flow which were published subsequently the most interesting are those devoted to analysis of stability of various types of laminar flows with respect to small perturbations of the free film surface [5–11]. The main conclusion following these papers is that all perturbations with sufficiently long (compared with the mean film thickness) wavelength grow when traveling downstream, the growth rate of such perturbations becoming higher with the increasing Reynolds number.

A remarkable contribution to the theory of wavy film flow is given by Shkadov [12] who proposed a new approach to analysis of film hydrodynamics based on a Fourier transformation method. By means of representation of the velocity profile in a form of a Fourier series it appears to be possible to devise a solution procedure different, in principle, from the conventional procedure of constructing a power series with respect to the small wave amplitude. However, within the framework of such a procedure two parameters from the four, namely the Reynolds number, the mean film thickness, the wavelength and the celerity, remain to be arbitrary, so the problem as in the case of the series expansion with respect to the small amplitude cannot be solved in a closed form without adoption of some additional physical hypotheses. As a remedy from this unfortunate situation the condition of the minimum film thickness at a given liquid flow rate was used in [12]. The resulting wavy regime (calculated in [12] for the case of wavelengths significantly exceeding the mean film thickness) was called 'the optimal wavy regime' on the basis of the fact that this regime, as follows from the stability analysis by means of the non-linear perturbation theory [13], is stable with respect to perturbations having the main parameters (the wavelength, the celerity and the amplitude) similar to those of the initial wavy flow. It should be noted however that the above mentioned criterion of "optimization" of wavy film flow is not substantiated from the theoretical point of view as the principle of minimum viscous dissipation of the kinetic energy is. In addition, a severe limitation of the method developed in [12] originates from use of the boundary layer equations for description of the velocity distribution in a film. It can be shown that the application to film flow of the system of the boundary layer equations is justified only within a very short range of the Reynolds numbers. Indeed let us consider the main equation which is easily derived from the system of the boundary layer equations and is generally used for the description of the wavy film flows [2, 12]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \left(\int \frac{\partial u}{\partial x}\,\mathrm{d}y\right)\frac{\partial u}{\partial y} = \frac{\sigma}{\rho}\frac{\partial^3 h}{\partial x^3} + \nu\frac{\partial^2 u}{\partial y^2} + g\,.$$

Dimensional analysis shows that the conditions of validity of this equation are:

$$(g\nu^4)^{\frac{1}{3}}\,(\rho/\sigma)\,Re^{\frac{5}{3}} \ll 1\,,$$

$$(g\nu^4)\ \ (\rho/\sigma)^{\frac{1}{4}}\,Re^{-}\ \ \ll 1\ .$$

These conditions should be satisfied simultaneously. The first condition means that the contribution of the inertial term $u(\partial u/\partial x)$ to the total momentum balance is comparable with the contribution of the surface tension term $(\sigma/\rho)\ \partial^3 h/\partial x^3$, both contributions being exceedingly large compared with the contribution omitted in the equation above) of the longitudinal molecular momentum transport term. The second condition means that the inertial $u(\partial u/\partial x)$ and the lateral molecular transport $\nu(\partial^2 u/\partial y^2)$ terms are of the same order of magnitude and both are exceedingly large compared with the neglected term $\nu(\partial^2 u/\partial x^2)$. The two conditions limit the range of the Reynolds numbers in which the boundary layer. approximation is valid. For example, this range appears to be $1 \lesssim Re \lesssim 20$ for water and $1 \lesssim Re \lesssim 7$ for ethanol.

Some papers were devoted to analysis of stationary regimes of film flow. In particular, the limiting case of extremely high Weber numbers was considered [14]. The attempt was made to take into account a non-parabolic form of the velocity distribution in a cross section of a film [15, 16]. A theory was developed concerning upward cocurrent flow of a wavy film and a turbulent gas in tubular conduits [17]. In the paper [18] a method was suggested for calculation of the parameters of wavy film flow in the case of arbitrary (not necessarily long compared with the mean film thickness) wavelengths. Using the linearized (with respect to the amplitude of waves) system of the Navier–Stokes equations the authors of the paper [18] substantiated a possibility of existence of a circulation cellular pattern of instantaneous streamlines in a wavy film. Such a pattern was predicted by Kapitza [2] on the basis of qualitative considerations. Subsequently the cellular system of closed streamlines was described quantitatively within the framework of the Kapitza's theory and the results of the calculation were used for interpretation of experimental data [19]. It should be noted, however, that the analysis given in [18] is based on an intrinsically contradictory system of equations and boundary conditions. This contradiction becomes obvious if one analyses the rigorous system of the Navier–Stokes equations and the corresponding boundary conditions for film flow. Let us consider a thin laminar liquid film thickness $h(x, t)$ descending under the action of the gravity force along a vertical plane $y = 0$. We shall assume that x-axis is directed downstream, the y-axis into the liquid. If the film is in contact with quiescent gas, and if the surface tension σ is constant along the film free surface, the velocity distribution inside the film should obey the following system of equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g\,,\quad(1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3)$$

with the boundary conditions at the free surface $y = h(x, t)$:

$$p + \frac{\sigma}{(1 + b^2)^{\frac{3}{2}}}\frac{\partial b}{\partial x} + \frac{2\rho\nu(1 + b^2)}{1 - b^2}\frac{\partial u}{\partial x} = 0, \quad (4)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \frac{4b}{1 - b^2}\frac{\partial u}{\partial x} = 0, \quad (5)$$

and at the solid surface $y = 0$:

$$u = v = 0, \quad (6)$$

where $b \equiv \partial h/\partial x$. The boundary conditions (4)–(5) are the conditions of balance of the normal and the tangential forces acting at the gas–liquid interface.† It follows from the equations (4) and (5) that in the linear approximation with respect to the dimensionless amplitude

$$A = (h_{max} - h_0)/h_0 \quad (7)$$

the boundary conditions for the normal and the tangential stresses at the free surface of a film should be written in the form:

$$p + \sigma\frac{\partial^2 h}{\partial x^2} - 2\rho\nu\frac{\partial u}{\partial x} = 0, \quad (8)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0. \quad (9)$$

In the paper [18] the corresponding conditions are written in the form: $p + \sigma(\partial^2 h/\partial x^2) = 0$, $(\partial u/\partial y) = 0$. From the dimensional analysis of the continuity equation (3) it follows, however, that $v \sim hu/\lambda$ and in the case $h \sim \lambda$ all the terms in the equations (8) and (9) are of the same order of magnitude. Consequently, the boundary conditions used in [18] are incorrect. Furthermore, in the case of arbitrary (not necessarily long compared with the mean film thickness) wavelength it is incorrect to omit any of the terms in the equation (2). Indeed, according to the equation (1) $\partial p/\partial x \sim \rho\nu(\partial^2 u/\partial x^2)$ (i.e. $p \sim \rho\nu u/\lambda$) and because of the relation $u \sim v$ the term $-\partial p/\rho\partial y$ in the equation (2) is of the same order of magnitude as all the other terms of this equation. Therefore there are no reasons for equating this term to zero separately as it is done in [18].

† The boundary conditions for mobile liquid–liquid and gas–liquid interfaces are formulated in a general form in [20]. In application to two-dimensional wavy film flow these conditions are written down in the papers [12, 21] but in both papers the explicit expressions for the normal and tangential components of the surface stress tensor contain some erroneous terms.

2. THE METHOD OF MOMENTS

In order to obtain the solution of the problem of wavy film flow we shall use the method analogous to the method of moments well known in the laminar boundary layer theory [22]. According to this method the local velocity profile $u_N(x, y, t)$ is approximated by a polynomial expression with respect to the lateral coordinate y:

$$u_N(x, y, t) = u_0 \sum_{k=1}^{N} a_k(\xi)\,(y/h_0)^k \; (N = 1, 2, \ldots), \quad (10)$$

where $\xi = (x - \alpha u_0 t)/h_0$, α—the dimensionless phase velocity of wave, u_0—the mean film velocity defined by the expression:

$$u_0 = \frac{1}{\lambda h_0}\int_0^{\lambda}\int_0^{h} u(\xi, y)\,d\xi\,dy. \quad (11)$$

The unknown functions $a_k(\xi)$ can be determined by means of the following procedure. Firstly the function $v_N(x, y, t)$ is found from the continuity equation (3):

$$v_N(x, y, t) = -u_0 \sum_{k=1}^{N} a_k'(\xi)\,(y/h)^{k+1}/k + 1, \quad (12)$$

where $a_k'(\xi) \equiv da_k/d\xi$. It is easily seen that the functions u_N and v_N automatically satisfy the no-slip boundary conditions (6). Then expressions (10) and (12) are inserted into the equation (2) and the latter is integrated with respect to over the interval from an arbitrary value of y to $y = h$. Such an integration permits, with the aid of the boundary condition (4) to obtain the pressure distribution $p(x, y, t)$. After insertion of the explicit expressions for the functions $p(x, y, t)$, $u_N(x, y, t)$ and $v_N(x, y, t)$ i to the equation (1) one arrives to the equation containing $N + 1$ unknown functions $a_1(\xi)$, $a_2(\xi)$, . . . , $a_n(\xi)$, $h(\xi)$. The closed system of $N + 1$ equations for determination of these functions can be obtained by successive multiplication of the equation (1) by 1, y, $y^2, \ldots, y^{N-2}$ and by integration of the results with respect to y in the limits from $y = 0$ to $y = h$. This integration procedure results in $N - 1$ ordinary differential equations. The remainder two equations are the boundary condition (5) and the condition of the microscopic mass balance in a film:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int u \mathrm{d}y = 0. \tag{13}$$

The resulting system of $N + 1$ ordinary differential equations can be solved for any given number N with the aid of modern high-speed digital computers.

3. THE LINEAR APPROXIMATION

It is interesting, however, firstly to investigate the problem by an approximate analytical method in order to elucidate qualitatively a role of the main controlled parameters of the system —the Reynolds and Weber numbers. Such an analytical investigation can be accomplished for the case of small wave amplitudes, ($A \ll 1$) when the solution of the problem can be sought by the traditional perturbation method. With the accuracy up to terms of the first order of smallness with respect to A and within the framework of the parabolic approximation of the velocity profile (10) with respect to $y(N = 2)$ the solution can be expressed through the only unknown function $\varphi(\xi)$:

$$u(\xi, y) = u_0[a_1(\xi)(y/h_0) + a_2(\xi)(y/h_0)^2], \tag{14}$$

$$v(\xi), y) = -u_0[a_1'(\xi)\,(y^2/2h_0^2) + a_2'(\xi)(y^3/3h_0^3)], \tag{15}$$

$$h(\xi) = h_0[1 + A\varphi(\xi)], \tag{16}$$

$$a_1(\xi) = 3 + 3(\alpha - 2)A\varphi(\xi) + \tfrac{1}{4}(3 - 2\alpha)A\varphi''(\xi), \tag{17}$$

$$a_2(\xi) = -\tfrac{3}{2} + \tfrac{3}{2}(3 - \alpha)A\phi(\xi) - \tfrac{3}{8}(3 - 2\alpha)A\varphi''(\xi). \tag{18}$$

By the above described procedure of inserting the expressions (14)–(18) into the equations (1), (2) and (4) and carrying out the appropriate integrations one can arrive to the following ordinary differential equation for the function $\varphi(\xi)$:

$$\frac{1}{A}\left(\frac{3}{Re} - \frac{gh_0}{u_0^2}\right) + 3(\alpha - 3)\frac{\varphi}{Re} - \left(\alpha^2 - \frac{12}{5}\alpha + \frac{6}{5}\right)\varphi' - \left(5\alpha - \frac{27}{4}\right)\frac{\varphi''}{Re} + \left(\frac{11}{40}\alpha^2 - \frac{219}{280}\alpha + \frac{153}{280} - \frac{1}{We}\right)\varphi''' + (\alpha - 3)\frac{\varphi^{\mathrm{IV}}}{40Re} - \left(\alpha^2 - \frac{39}{14}\alpha + \frac{27}{14}\right)\frac{\varphi^{\mathrm{V}}}{80} + (3 - 2\alpha)\frac{\varphi^{\mathrm{VI}}}{160Re} = 0. \tag{19}$$

The necessary condition for existence of a periodic solution of the equation (19) satisfying the condition $\overline{\phi(\xi)} = 0$ (the line above means averaging the wavelength) is

$$h_0^3 = \frac{3v^2 Re}{g}. \tag{20}$$

This condition determines the mean film thickness at a given value of the Reynolds number. If

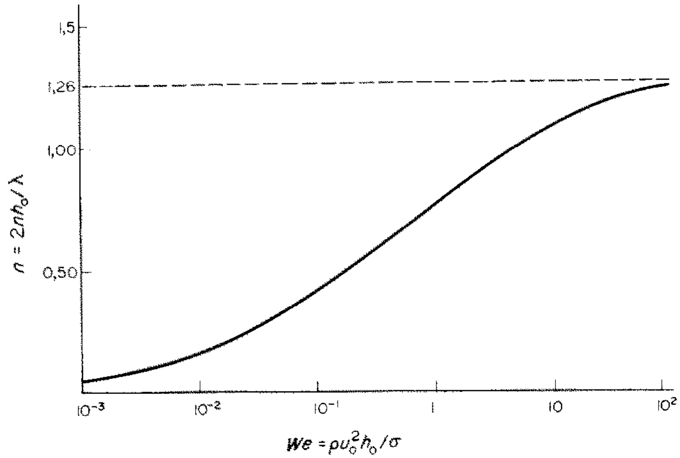

FIG. 1. Dependence of the dimensionless wave number n on the Weber number in the linear approximation with respect to the parameter A.

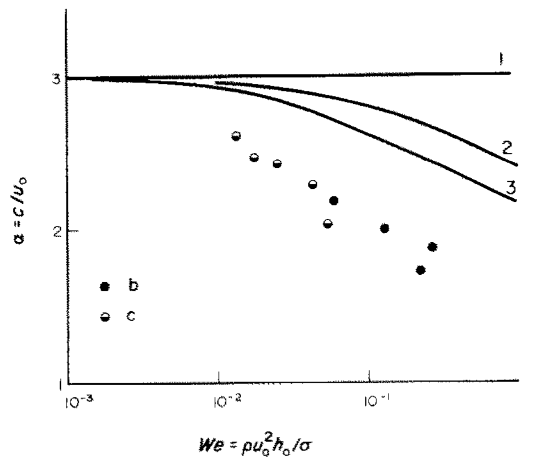

FIG. 2. The dimensionless phase velocity of wave α as a function of the Weber number in the linear approximation with respect to the parameter A: 1—the result of the Kapitza's theory [2]; 2—the theoretical curve obtained in the work [18]; 3—the curve obtained in our work; the experimental data; b—water [18], c—alcohol [3].

the condition (20) is satisfied one can seek the solution of the equation (19) in the form:

$$\varphi(\xi) = \sin(n\xi), \qquad (21)$$

where $n = 2\pi h_0/\lambda$ is the dimensionless wave number. Inserting the expression (21) into the equation (19) and equating the coefficients at $\sin(n\xi)$ and $\cos(n\xi)$ one obtains the following system of algebraic equations for n and α:

$$n^4\left(\alpha^2 - \frac{39}{14}\alpha + \frac{27}{14}\right) + 2n^2\left(11\alpha^2 - \frac{219}{7}\alpha - \frac{153}{7} - \frac{40}{We}\right) + 80\left(\alpha^2 - \frac{12}{5}\alpha + \frac{6}{5}\right) = 0, \qquad (22)$$

$$n^6(2\alpha - 3) + 4n^4(\alpha - 3) + 160n^2\left(5\alpha - \frac{27}{4}\right) + 480(\alpha - 3) = 0. \qquad (23)$$

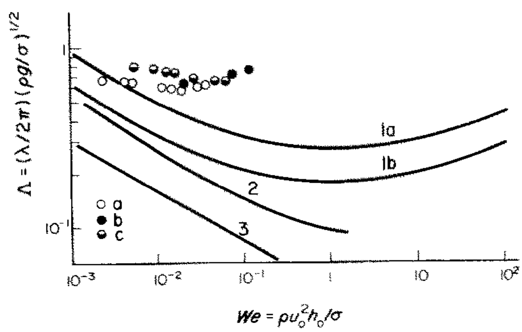

FIG. 3. The dimensionless wavelength Λ as a function of the Weber number (in the linear approximation with respect to the parameter A) at the fixed values of the surface tension, the density and the viscosity: 1a—$\sigma = 23$ dynes/cm, $\rho = 0{\cdot}8$ g/cm^3, $\nu = 1{\cdot}38 \,.\, 10^{-2}$ cm^2/s; 1b—$\sigma = 72$ dynes/cm, $\rho = 1{\cdot}0$ g/cm^3, $\nu = 1{\cdot}00 \,.\, 10^{-2}$ cm^2/s; 2—the curve obtained in the work [18]; 3—the result of the Kapitza's theory [2]; the experimental data: a—water [3], b—water [18], c—alcohol [3].

This system significantly differs from the corresponding system obtained in the paper [18]. The origin of the difference was discussed earlier. In the Figs. 1–3 the results of numerical solution of the system (22) and (23) are represented graphically. In particular, it is seen that in accordance with the available experimental data the values n and α are monotonous functions of the Weber number. In the limiting case $We \to \infty$ we obtain $n = 1{\cdot}26$; $\alpha = 1{\cdot}80$. The corresponding results of the paper [18] are $n = 4{\cdot}42$ $\alpha = 1{\cdot}69$.

4. WAVES OF A FINITE AMPLITUDE

The linear approximation provides only qualitative description of the real wavy pattern. Besides, within the linear approximation as well as within any approximation which is obtained by means of expansion of the solution into a power series with respect to the parameter A this parameter remains indefinite. For calculation of the parameters of waves with a finite amplitude the modified Galerkin's method may be used.† According to this method the velocity distribution is represented in the form of the polynomial expressions (14) and (15). The functions $a_1(\xi)$, $a_2(\xi)$ and $h(\xi)$ are expanded into the Fourier series with respect to ξ and the expansions are approximated by a finite number of terms:

$$a_1(\xi) = a_{10} + a_{11}\sin(n\xi) + \bar{a}_{11}\cos(n\xi) + \ldots + a_{1j}\sin(jn\xi) + \bar{a}_{1j}\cos(jn\xi)\,, \qquad (24)$$

$$a_2(\xi) = a_{20} + a_{21}\sin(n\xi) + \bar{a}_{21}\cos(n\xi) + \ldots + a_{2j}\sin(jn\xi) + \bar{a}_{2j}\cos(jn\xi)\,, \qquad (25)$$

$$h(\xi) = h_0\,[1 + A\sin(n\xi) + b_2\sin(2n\xi) + b_2\cos(2n\xi) + \ldots + b_j\sin(jn\xi) + b_j\cos(jn\xi)]\,. \qquad (26)$$

The total number of the unknown quantities in the expressions (24)–(26) is $bj + 4$. These quantities are a_{ik}, $\bar{a}_{ik}(i = 1,2;\, k = 0,1 \ldots j)$, $b_i(i = 2, 3, \ldots, j)$, h_0, n, α and A. For determination of these quantities the following procedure is suggested. The expressions (14) and (15) together with the expansions (24)–(26), are inserted into the equation (2) and the latter is integrated with respect to y over the interval from an arbitrary y to $y = h$. Then the boundary condition is used and the pressure distribution is found. This distribution, together with expressions (14), (15), (24)–(26), is inserted into the equation (1) and after integration over the interval from $y = 0$ to $y = h$ the final equation is obtained the left side of which may be represented as a truncated Fourier series. Equating to zero the coefficients at the first j harmonics of this series and using the Newton's tangent method it is possible to obtain $2j + 1$ relationships between the quantities a_{1i}, $\bar{a}_1{}^i$, a_{2i}, $\bar{a}_{2i}$, b_i, $\bar{b}_i$, h_0, α, n and A. Similarly $2(2j + 1)$ more relationships may be derived from the boundary condition (5) and from the macroscopic mass balance condition (13). Therefore in order to determine all $6j + 4$ characteristics of the wavy film flow it is necessary to get one more additional physical condition.† The Reynolds stability principle [23] may play the role of such an auxiliary condition. According to this principle the flow regime is stable with respect to any (not necessarily infinitesimal) perturbation of the kinetic energy of this perturbation does not grow with time, i.e. if

$$\frac{\mathrm{d}E'}{\mathrm{d}t} \leqslant 0\,. \qquad (27)$$

Let us consider as an initial flow regime the wavy regime characterized by the vector velocity distribution $\boldsymbol{v}(A)$ where A is the wave amplitude defined by the expression (26) (the temporal and the special independent variables of the velocity distribution we shall omit for brevity). The definite kinetic energy distribution

† In application to wavy film flows this method, within the framework of the boundary-layer approximation, was firstly proposed in the paper [12].

† It should be kept in mind that such a necessity arises solely due to the accepted form of the solution as a set of the truncated Fourier expansions (24)–(25). In a case of using the infinite series there would be no need in any additional conditions for solving the problem.

rate dE/dt corresponds to the initial regime. This rate is a functional of the velocity distrubition $\mathbf{v}(A)$ and is determined by the expression:

$$\frac{dE}{dt} \equiv \Phi[\mathbf{v}(A)] = \frac{\rho}{2}\int_0\int_0 \frac{\partial \mathbf{v}^2}{\partial t}\,dy dx$$

$$= \int_0^\lambda\int_0^h \left\{\rho g u - \frac{\partial}{\partial x}\left(u\left[\frac{\rho(u^2+v^2)}{2}+p\right]\right) - \frac{\partial}{\partial y}\left(v\left[\frac{\rho(u^2+v^2)}{2}+p\right]\right) + \rho v(u\Delta u + v\Delta v)\right\} dy dx. \quad (28)$$

Let us suppose that the initial flow is disturbed by some external infinitesimal perturbation so that in a disturbed state the amplitude A acquires the infinitesimal increment ΔA:

$$A^* = A + \Delta A. \quad (29)$$

In this case the resulting rate of the kinetic energy dissipation, dE^*/dt, may be represented in a form of the following expansion:

$$\frac{dE^*}{dt} = \Phi[\mathbf{v}(A)] + \frac{d\Phi}{dA}[\mathbf{v}(A)]\,\Delta A + 0(\Delta A^2). \quad (30)$$

Therefore in the case of an infinitesimal perturbation of the initial regular wavy flow regime the stability condition (27) takes the form:

$$\frac{dE'}{dt} = \frac{d\Phi}{dA}[\mathbf{v}(A)]\,\Delta A + 0(\Delta A^2) \leqslant 0. \quad (31)$$

It follows from the condition (31) that for existence of the stable wavy film flow characterized by the amplitude A it is sufficient that the condition

$$\frac{d\Phi}{dA}[\mathbf{v}(A)] = 0 \quad (32)$$

be satisfied. It is seen from the equations (28) and (32) that in the limiting case of flow with the extremely low Reynolds numbers when the inertial terms in the equation (28) can be neglected compared with the viscous terms the condition (32) becomes to be equivalent to the principle of minimum of the viscous kinetic energy dissipation [2].

The procedures of expansion of the functions $a_1(\xi)$, $a_2(\xi)$ and $h(\xi)$ into truncated Fourier series and determination, taking into account the condition (32), of the coefficients of the truncated series as well as the parameters n and α were programmed for a digital computer. Calculations were performed with the accuracy corresponding to keeping two first harmonics in the expansions (24)–(26), i.e. to $j = 2$. The Figs. 4–6 show the results of numerical calculation of the parameters n, α and A for a water film in the range of the Reynolds numbers from $Re = 1$ to $Re = 40$ (correspondingly from $We = 10^{-3}$ to $We = 10^{-1}$. At a fixed value of the surface tension σ the dimensionless wavelength $\Lambda = (\lambda/2\pi)\,(\rho g/\sigma)^{\frac{1}{2}}$ is a non-monotonous function of the Weber number (and correspondingly of the Reynolds number because at a fixed value of the surface tension $We = \text{const.}\ Re^{\frac{5}{3}}$) This function appears to be different for different liquids and passes through a minimum with increasing flow rate[4]. It is seen from the figures that the calculation results are in rather good agreement with the available experimental data.

5. EFFECT OF WAVES ON FILM MASS TRANSFER

Knowledge of hydrodynamics of wavy film flow permits to analyse in detail the effect of waves on film mass transfer. In some papers [2, 18, 19] attention was paid to the fact that at sufficiently big amplitudes of waves the mean value of the x-component of the velocity, i.e.

$$\bar{u}(\xi) = \frac{1}{h_0}\int u(\xi, y)\,dy = u_0[a_1(\xi)(h^2/2h_0^2) + a_2(\xi)(h^3/3h_0^3)] \quad (33)$$

becomes zero at the definite cross-section of the film. The planes $\xi = \text{const}$ in which the condition $\bar{u}(\xi) = 0$ is satisfied separate the regions where the instantaneous pattern of stream lines reminds a "circulation cell" [18]. each of these regions the streamlines are closed. According to our results the lowest limit of the Reynolds

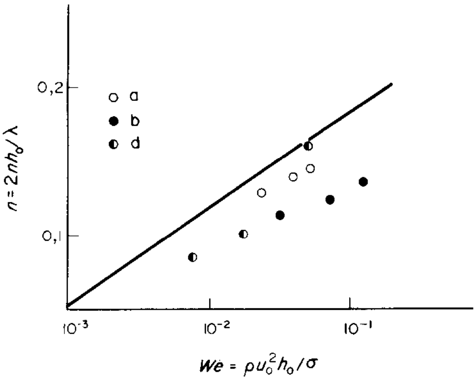

FIG. 4. Dependence $n(We)$ for water calculated by a digital computer (solid line); the experimental data: a—[3], b—[18] d—[24].

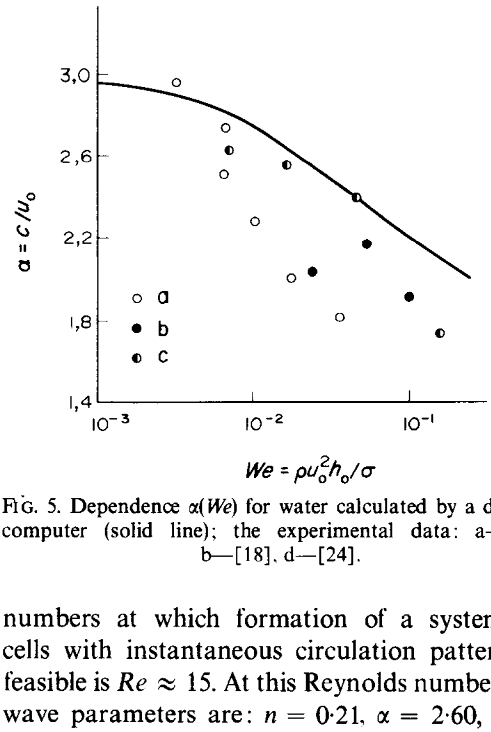

FIG. 5. Dependence $\alpha(We)$ for water calculated by a digital computer (solid line); the experimental data: a—[3], b—[18], d—[24].

numbers at which formation of a system of cells with instantaneous circulation pattern is feasible is $Re \approx 15$. At this Reynolds number the wave parameters are: $n = 0{\cdot}21$, $\alpha = 2{\cdot}60$, $A = 0{\cdot}39$. On the basis of the fact mentioned it was concluded that liquid elements may leave surface regions of the film and get to bulk regions, herein there may exist so-called "renewal" of the free film surface intensifying mass transfer. It can be shown, however, that such interpretation is incorrect. Indeed, a consideration dealing with the velocities u and v which are defined in the stationary coordinate system ($u = v = 0$ at $y = 0$) permits to get information only about instantaneous (corresponding to some fixed moment of time) hydrodynamic pattern of film flow. Such a consideration does not put one in a position to

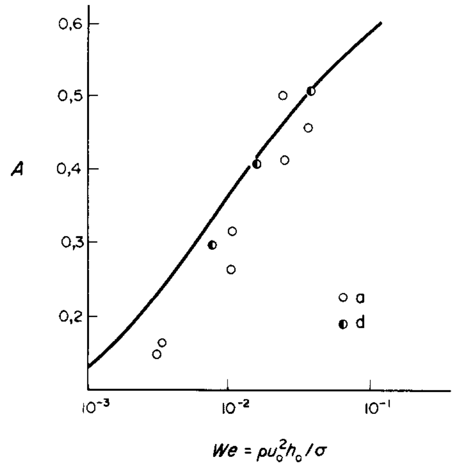

FIG. 6. Dependence $A(We)$ for water calculated by a digital computer (solid line); the experimental data: a—[3], d—[24].

assert about existence of liquid circulations because with time going on the liquid elements must not only displace themselves to other parts of their instantaneous trajectories but also must take part in translational motion along the x-axis with the velocity αu_0. In order to clarify the question about possibility for liquid elements to leave the interface and to move into deeper regions of the film it is necessary to determine the trajectories of the liquid elements in the coordinate system travelling downstream with the velocity. When using this coordinate system the time will not explicitly enter the motion equations and consequently the streamlines will coincide with the trajectories of the liquid elements. In a differential form the equation for the trajectories in the above defined relative coordinate system is

$$\frac{1}{h_0}\frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{a_1'(\xi)\,(y^2/2h_0^2) + a_2'(\xi)\,(y^3/3h_0^3)}{\alpha - a_1(\xi)\,(h/h_0) - a_2(\xi)\,(y^2/h_0^2)}. \tag{34}$$

The denominator in the right side of the equation (34) nowhere becomes zero because according to our results the condition $u(\xi, y) < \alpha$ is satisfied throughout all the space occupied by a film. It means that the trajectories of liquid elements are unclosed. The trajectories would be closed only if the condition $u(\xi, y) \geqslant \alpha$ took place at some distance from the plane $y = 0$. Therefore the liquid elements taking part in a periodic wavy motion inside a falling film travel along unclosed trajectories. The equation for these trajectories can be obtained by integration of the equation (34). After integration we arrive to the following form of the trajectories:

$$a_1(\xi)(y^2/2h_0^2) + a_2(\xi)(y^3/3h_0^3) - \alpha(y/h_0) = \text{const}. \quad (35)$$

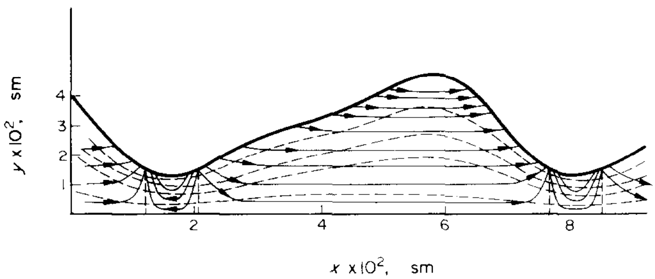

FIG. 7. The solid lines with arrows—instant streamlines. The dotted lines—liquid particle trajectories in coordinates translating along the film length with velocity $\alpha u_0 (\alpha = 1{\cdot}98)$ with respect to the solid wall.

This is a family (Fig. 7) of tortuous unclosed curves rarefying under the crests of waves and condensing in the dents. Thus the foregoing analysis shows that under conditions of regular wavy film flow there can be no "renewal" of the free surface by fresh portions of liquid coming from the interior. The effect of waves on mass transfer should exhibit itself mainly in increasing of the area of the mass transfer surface as well as in augmentation of convective transport due to condensation of streamlines in the wave dents. It should be noted, however, that the flow pattern postulated in solving the system (1)–(6) and assuming the same celerity for αu_0 all wavelengths may not exist in real film systems. More detailed description of wavy film flow should be based on taking into account a discrete spectrum of celerities. Within the framework of such a detailed flow pattern it is feasible, in principle, to account for formation of circulation currents of liquid elements in a film. In this situation the film will show some resemblance with a system of droplets rolling along the solid surface. Liquid circulation in such a system may cause significant augmentation of mass transfer.

6. CONCLUSIONS

(1) All previous papers on a theory of wavy film flow are based either on incorrect systems of equations and boundary conditions or on the boundary layer approximation, the latter being justified only in a limited range of the Reynolds numbers. The fully self-consistent description of wavy film flow should be based on a complete system of the Navier–Stokes equations with corresponding boundary conditions.

(2) From the linear approximation with respect to the ratio of the wave amplitude to the mean film thickness it follows that the wavelength and the celerity are functions of the Reynolds and Weber numbers. At fixed values of the density, the surface tension and the viscosity of liquid the celerity decreases monotonously with the increasing flow rate, whereas the wavelength as a function of the flow rate passes through minimum.

(3) The results of numerical calculation of the wavy regime parameters obtained with the aid of the modified Galerkin's method (taking into

account the Reynolds stability principle) are in good agreement with the available experimental data. This agreement evidences, in particular, that the distribution of x-velocity components may be approximated by a parabolic function of y with coefficients expandable into the harmonic Fourier series with respect to $n\xi$.

(4) Under conditions of regular wavy film flow liquid elements inside the film travel along unclosed trajectories and there can be no "renewal" of the film surface by fresh portions of liquid coming from the interior of the film. The effect of waves on film mass transfer exhibits itself mainly in increasing of the free surface area and augmentation of convective mass transfer due to condensation of liquid streamlines in the wave dents.

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WELLIGE STRÖMUNG IN DÜNNEN FLÜSSIGKEITSFILMEN

Zusammenfassung—Es wird eine integrale Methode zur Lösung des Systems der Navier–Stokesschen Gleichungen entwickelt, für die Geschwindigkeitsverteilung in einem laminaren Flüssigkeitsfilm der unter dem Einfluss der Schwerkraft frei an einer senkrechten festen Wand fliesst. Mit Hilfe dieser Methode werden die charakteristischen Parameter der regulären Filmströmung (Wellenlänge, Geschwindigkeit, Amplitude) berechnet. Bei der linearen Approximation in Bezug auf das Verhältnis von Amplitude zu mittlerer Filmdicke erhält man analytische Ausdrücke, die die Abhängigkeit der Wellenlänge und der Geschwindigkeit von der Weberzahl bestimmen. Die Parameter von Wellen, deren Amplitude mit der mittleren Filmdicke vergleichbar ist, werden mit einem Digitalrechner ermittelt. In Übereinstimmung mit verfügbaren experimentellen Daten zeigt sich, dass für gegebene Werte der Oberflächenspannung, der Dichte und der Viskosität der Flüssigkeit die Geschwindigkeit mit steigender Weberzahl monoton fällt, während die Wellenlänge als Funktion der Weberzahl ein Minimum aufweist. Der Mechanismus des Einflusses der Wellen auf den Stoffübergang wird diskutiert.

ECOULEMENT DE MINCES FILMS LIQUIDES AVEC ONDES

Résumé—On développe une méthode intégrale pour résoudre le système des équations de Navier-Stokes qui décrit la distribution de vitesse dans un film liquide laminaire s'écoulant librement par gravité le long d'une paroi solide verticale. On a calculé à l'aide de cette méthode les paramètres caractéristiques de l'écoulement régulier du film (la longueur d'onde, la célérité et l'amplitude). Dans l'approximation linéaire qui considère le rapport de l'amplitude à l'épaisseur moyenne du films, on a obtenu des expressions analytiques qui déterminent la dépendance de la longueur d'onde et de la célérité au nombre de Weber. On linéaire qui considère le rapport de l'amplitude à l'épaisseur moyenne du film, on a obtenu des expressions moyenne du film. En accord avec les résultats expérimentaux disponibles il est trouvé qu'à des valeurs données de la tension superficielle, de la densité et de la viscosité du liquide, la célérité décroît de façon monotone lorsque le nombre de Weber augmente, tandis que la longueur d'onde passe par un minimum. On discute le mécanisme de l'effet des ondes sur le transfert massique.

ВОЛНОВОЕ ТЕЧЕНИЕ ТОНКИХ ЖИДКИХ ПЛЁНОК

Аннотация—Разработан интегральный метод решения системы уравнений Навье-Стокса для распределения скорости в ламинарной жидкой пленке, свободно текущей под действием гравитационной силы вдоль вертикальной твердой стенки. С помощью этого метода рассчитаны характеристические параметры (длина волны, скорость и ампдитуда) равномерного пленочного течения. Получены аналитические зависимости длины волны и скорости от числа Вебера в линейном приближении по отношению амплитуды к средней толщине пленки. На вычислительной машине рассчитаны параметры волн с амплитудой, сравнимой со средней толщиной пленки. Согласно имеющимся экспериментальным данным найдено, что для данных значений поверхностного натяжения, плотности и вязкости жидкости скорость монотонно уменьшается с увеличением числа Вебера, причем длина волны как функция числа Вебера проходит через минимум. Обсуждается механизм влияния волн на массоперенос.